UNIT-4

# Z- Transform

## 4.14.1 Advantages of Z transform

1. Discrete time signal & LTI systems can be completely characterized using Z transform
2. Stability of LTI system can be determined
3. Mathematical calculations ~~can~~ are reduced using Z transform.
4. Soln of differential eq'ns can be simplified using Z transform.

There are basically two types of Z transform

Two types → ① Single sided Z transform
② Double " "

$$X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n} \quad \text{——} (4.30)$$

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{——} (4.31)$$

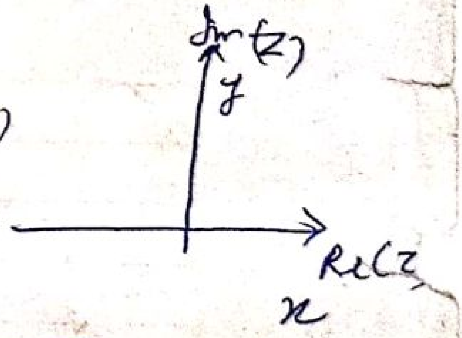



Figure Z domain

Here Z is complex variable

ROC → Region of Convergence → The region of convergence of X(Z) is set for all the values of Z for which X(Z) attains a finite value. Every- when we find the Z transform.

## 4.14.3 Significance of ROC →



(1) ROC will decide whether system is stable or not

(2) ROC also determines ~~type~~ type of sequence means

(i) Causal or non causal (ii) finite or infinite

**Example 4.18**

# obtain Z transform

$x(n) = [1, 2, 8, 6, 0, 7, 1]$
$\uparrow$

Standard formula of Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$x(0) = 1$ $\quad$ $x(3) = 6$ $\quad$ $x(6) = 1$

$x(1) = 2$ $\quad$ $x(4) = 0$

$x(2) = 8$ $\quad$ $x(5) = 7$

$$X(Z) = \sum_{n=0}^{6} x(n) z^{-n}$$

$$X(Z) = x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5} + x(6) z^{-6}$$

$$= 1 \cdot z^{0} + 2z^{-1} + 8z^{-2} + 6z^{-3} + 7 \times z^{-5} + 1 \times z^{-6}$$

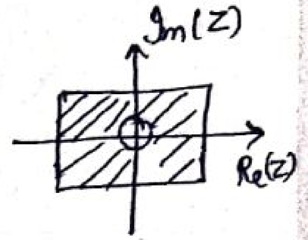
$$X(Z) = 1 + \frac{2}{z} + \frac{8}{z^2} + \frac{6}{z^3} + \frac{7}{z^5} + \frac{1}{z^6}$$

To find ROC put $Z = 0$ and $Z = \infty$

$$X(Z) = 1 + \frac{2}{0} + \frac{8}{0} + \frac{6}{0} + \frac{7}{0} + \frac{1}{0} = 1 + \infty = \infty$$

$$X(Z) = 1 + \frac{2}{\infty} + \frac{8}{\infty} + \frac{6}{\infty} + \frac{7}{\infty} + \frac{1}{\infty} = 1 + 0 = 1$$

The ROC is entire Z plane except $|Z| = 0$




Example 4.19 = $\{1, 2, 8, 6, 0, 7, \underset{\uparrow}{1}\}$

The standard formula of Z transform is

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$x(0) = 1$ $\quad$ $x(-4) = 8$

$x(-1) = 7$ $\quad$ $x(-5) = 2$

$x(-2) = 0$ $\quad$ $x(-6) = 1$

$x(-3) = 6$

$$X(Z) = \sum_{n=0}^{-6} x(n) Z^{-n}$$

$$X(Z) = x(0) Z^0 + x(-1) Z^1 + x(-2) Z^2 + x(-3) Z^3 + x(-4) Z^4 + x(-5) Z^5 + x(-6) Z^6$$

$$= 1 \cdot Z^0 + 7 \cdot Z^1 + 0 \cdot Z^2 + 6 Z^3 + 8 Z^4 + 2 Z^5 + 1 Z^6$$

$$= 1 + 7Z + 0 + 6Z^3 + 8Z^4 + 2Z + 1Z^6$$

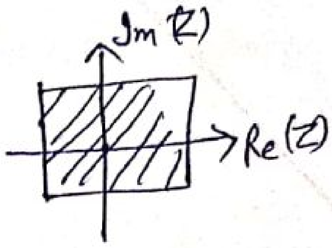
To find ROC Put $Z = 0$ and $Z = \infty$

If put $Z = 0$ = $1 + 0 + 0 + 0 + 0 + 0 + 0 = 1$

If Put $Z = \infty$ = $1 + \infty + \infty + \infty + \infty + \infty + \infty = 1 + \infty = \infty$

The ROC is entire Z plane except $|Z| = \infty$

**Example 4.20** = $\{1, 2, \underset{\uparrow}{8}, 6, 0, 7, 1\}$




The standard formula of Z transform is

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$x(-2) = 1$ $\quad$ $x(2) = 0$

$x(-1) = 2$ $\quad$ $x(3) = 7$

$x(0) = 8$ $\quad$ $x(4) = 1$

$x(1) = 6$

$$\sum_{n=-2}^{4} x(n) Z^{-n}$$

$$X(Z) = x(-2)Z^{2} + x(-1)Z^{1} + x(0)Z^{0} + x(1)Z^{-1} + x(2)Z^{-2} + x(3)Z^{-3} + x(4)Z^{-4}$$

$$= 1Z^{2} + 2Z^{1} + 8Z^{0} + 6Z^{-1} + 0Z^{-2} + 7Z^{-3} + 1Z^{-4}$$

$$X(Z) = 1Z^{2} + 2Z + 8 + \frac{6}{Z} + 0 + \frac{7}{Z^{3}} + \frac{1}{Z^{4}}$$

To find ROC put $Z = 0$ and $Z = \infty$ $\left\{ \begin{array}{l} \text{entire Z plane} \\ \text{except } |Z| = 0 \text{ and} \\ |Z| = \infty \end{array} \right.$

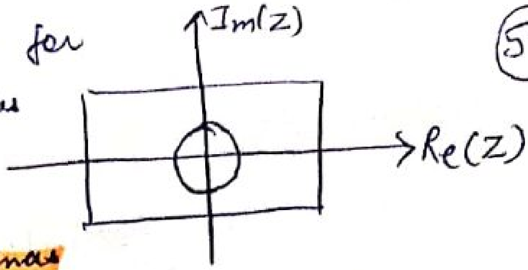
Put $Z = 0$

$$X(Z) = 0 + 0 + 8 + \frac{6}{0} + 0 + \frac{7}{0} + \frac{1}{0}$$

$$0 + 0 + 8 + \infty + 0 + \infty + \infty = 8 + \infty = \infty$$

Put $Z = \infty \Rightarrow \infty + \infty + 8 + 6/\infty + 0 + 7/\infty + 8/\infty = \infty + 0 = \infty$

Sequence is both sided. bcs $x(n)$ is present for both +ve & -ve values of $n$. Thus for both sides finite duration sequence the ROC is entire Z plane except $|z|=0$ and $|z|=\infty$



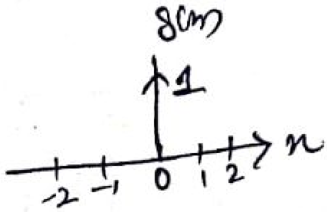

### 4.14.4 Z Transform of Standard Sequences

**Example** → Find Z transform of unit Impulse function $\delta(n)$

Solution

$x(n) = \delta(n)$

$$\delta(n) = \begin{cases} 1 & \text{for } n=0 \\ 0 & \text{for } n \neq 0 \end{cases}$$




The standard formula of Z transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} \delta(n) Z^{-n}$$

$$X(Z) = \delta(0) Z^{0} \Big|_{\text{at } n=0}$$

$$X(Z) = 1 \cdot 1$$

$$X(Z) = 1$$

There is no 'Z' term, So ROC is entire Z plane

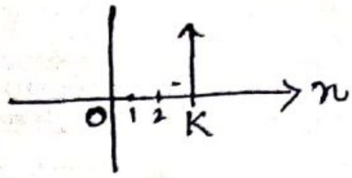
That means Z can have any value.

Example #4.22 Find Z transform of shifted impulse for $\delta(n-k)$



$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \qquad x(n) = \delta(n-k)$$

$$X(z) = \delta(n-k)\, z^{-k}$$




$\delta(n-k) = 1$ only at $n = k$)

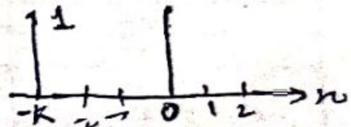
only at $n = k$ its values is 1

$\frac{1}{z^k}$

$X(z) = z^{-k}$ [ROC is entire z plane except $z=0$]

Example #4.23

$$\delta(n+k) = z^{k}$$



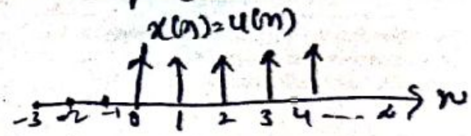

$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} \Rightarrow X(z) = \delta(n+k) z^{-k}$

$= \delta(n+k) = 1$ only at $n = k \Rightarrow X(z) = z^{k}$

only at $n = -k$ its values is 1

# $u(n)$ Example 4.24 find the Z transforms of unit step function

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$




$$\sum_{n=0}^{\infty} u(n)\, z^{-n} \qquad \sum_{n=0}^{\infty} 1\, z^{-n}$$

$$\sum_{n=0}^{\infty} (z^{-1})^{n}$$

Stand formula

$$\sum_{n=0}^{\infty} A^{n} = \frac{1}{1-A} \quad \text{if } |A| < 1$$

$$\frac{1}{1-z^{-1}} \qquad |z^{-1}| < 1$$

ROC $|z^{-1}| < 1$

$\left|\frac{1}{z}\right| < 1$

$|1| < |z|$

ROC $|z| > 1$

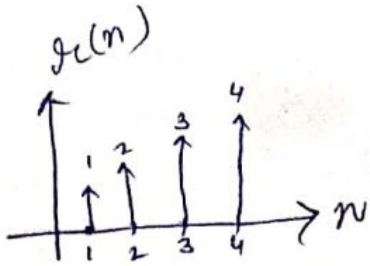

exterior part of circle having radius

$$\frac{1}{1-\frac{1}{z}} \qquad \frac{1}{\frac{z-1}{z}} \qquad \frac{z}{z-1} \quad \text{if } |z^{-1}| < 1$$

Example 4.25

## Z transform of unit ramp →






at $n=0 \quad r(n)=0$

$n=1 \quad r(n)=1$

$n=2 \quad r(n)=2$

$n=3 \quad r(n)=3$

$n=4 \quad r(n)=4$

$r(n) = n$ for $n \geq 0$

$0$ otherwise

$r(n) = n\,u(n)$

Standard formula of Z transform is

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \Rightarrow \sum_{n=-\infty}^{\infty} n\,u(n) Z^{-n}$$

$$\sum_{n=0}^{\infty} n Z^{-n} \Rightarrow \sum_{n=0}^{\infty} n (Z^{-1})^{n} \Rightarrow \sum_{n=0}^{\infty} n A^{n} = \frac{A}{(1-A)^2}$$

Standard formula $\quad |A|<1$

where $A = Z^{-1}$

$$X(Z) = \frac{Z^{-1}}{(1-Z^{-1})^2} \quad \text{if } |Z^{-1}|<1$$

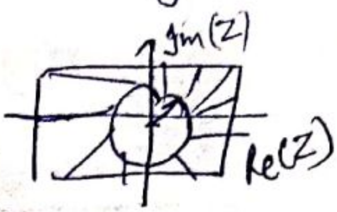
To convert this eq'n into the power of 'Z' we will multiply numerator and denominator by $Z^2$

$$\frac{Z^2 Z^{-1}}{Z^2(1-Z^{-1})^2} \Rightarrow \frac{Z}{Z^2(1+Z^{-2}-2Z^{-1})} \Rightarrow$$

$$\frac{Z}{Z^2-2Z+1} \quad \text{if } |Z^{-1}|<1$$

$$X(Z) = \frac{Z}{(Z-1)^2} \quad \text{if } |Z^{-1}|<1$$

$|Z| > 1$ ROC is exterior part of circle having radius 1.

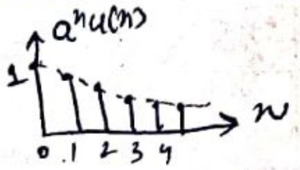


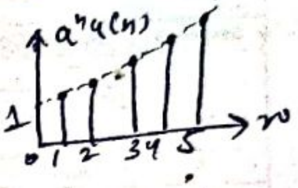

$$n\,u(n) \longleftrightarrow \frac{Z}{(Z-1)^2} \quad \text{if } |Z| > 1$$

Example 4.28

## # Z transform of right hand exponential sequence →

$x(n) = \alpha^n u(n)$




$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \Rightarrow \sum_{n=0}^{\infty} \alpha^n Z^{-n}$$

$$X(Z) = \sum_{n=0}^{\infty} (\alpha Z^{-1})^n \Rightarrow \frac{1}{1-A} \quad |A| < 1 = \sum_{n=0}^{\infty} A^n$$

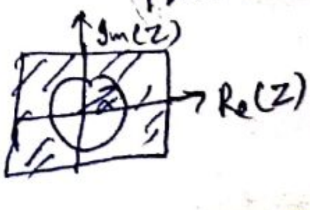
$$\frac{1}{1-\alpha z^{-1}} \qquad |\alpha z^{-1}| < 1$$

To convert +ve power of Z multiply & divide by Z

$$\frac{Z}{Z-\alpha} \quad \text{if } |\alpha Z^{-1}| < 1 \Rightarrow \left|\frac{\alpha}{Z}\right| < 1$$

$$|\alpha| < |Z|$$

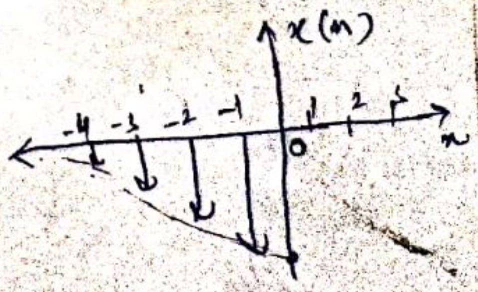
exterior part of circle having radius $\alpha$.




Example

## Z transform of left hand exponential sequence →

$x(n) = -\alpha^n u(-n-1)$




$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$x(n) = -\alpha^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-\alpha^n) z^{-n} \Rightarrow \sum_{n=-\infty}^{-1} -(\alpha^1 z^{-1})^n$$

The limit of summation are -ve, so to make it +ve we will substitute

$$l = -n \qquad n = -l$$

$$n = -\infty \Rightarrow +\infty = +l \quad = l = \infty$$
$$n = -1 \Rightarrow -1 = -l \quad = l = 1$$

$$X(z) = + \sum_{l=1}^{\infty} -(\alpha^1 z^{-1})^{-l}$$

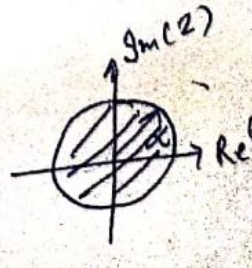
$$X(z) = -\sum_{l=1}^{\infty} (\alpha^{-1} z^1)^l \Rightarrow \sum_{l=1}^{\infty} A^l = A \times \frac{1}{1-A} \quad |A|<1$$

$$A = \alpha^{-1} z$$

$$X(z) = -(\alpha^{-1} z) \times \frac{1}{1-\alpha^{-1} z} \qquad |\alpha^{-1} z| < 1$$

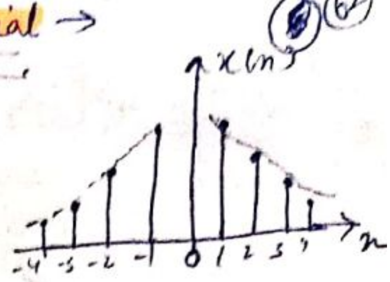
$$-\frac{z/\alpha}{1-z/\alpha} \quad \left|\frac{z}{\alpha}\right| < 1 \Rightarrow \frac{z/\alpha}{z/\alpha - 1} \Rightarrow |z| < |\alpha|$$

$$\frac{z}{z-\alpha} \qquad |z| < |\alpha|$$




ROC interior part of circle having radius α

## Z- transform of two sided exponential →




$$\alpha^n u(n) + \beta^n u(-n-1)$$

$$\alpha^n u(n) \rightarrow \frac{Z}{Z-\alpha} \quad |Z| > |\alpha|$$

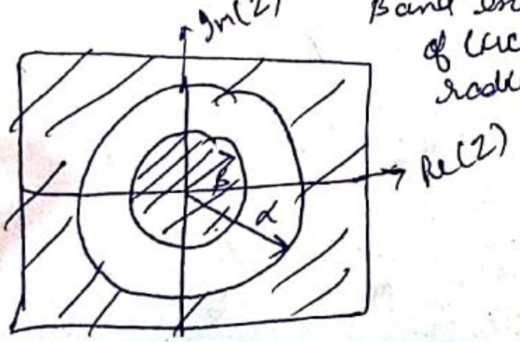
$$\beta^n u(-n-1) \rightarrow -\frac{Z}{Z-\beta} \quad |Z| < |\beta|$$

$$\frac{Z}{Z-\alpha} - \frac{Z}{Z-\beta} \quad |\beta| > |Z| > |\alpha|$$

Case 1

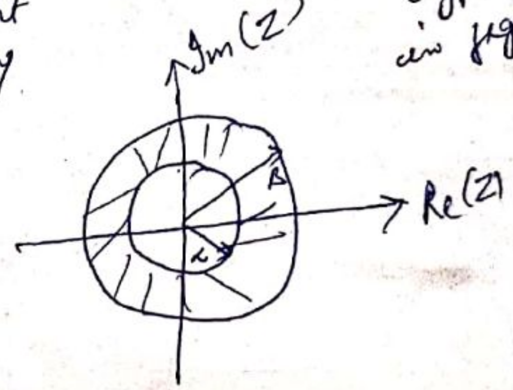
$|\beta| < |\alpha|$

ROC is defined for interior part of circle having Radius β and exterior part of circle having radius α




Case 2 $|\beta| > |\alpha|$

ROC is Ring type shown in figure




## 4.15 Properties of Z transform →

Linearity → $x(n) = a_1 x_1(n) + a_2 x_2(n)$

$x_1(n) \overset{Z}{\longleftrightarrow} X_1(Z) \qquad x_2(n) \longleftrightarrow X_2(Z)$

$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$

Proof → according to definition of Z transform

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] Z^{-n}$$

$$X(Z) = \sum_{n=-\infty}^{\infty} a_1 x_1(n) Z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) Z^{-n}$$

$$X(Z) = a_1 \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n}$$

$$X(Z) = a_1 X_1(Z) + a_2 X_2(Z)$$

ROC → The combined ROC is the overlap of the individual ROC's of $X_1(Z)$ & $X_2(Z)$

# Determine Z transform & ROC of signal

Example 4.21

$$x(n) = [3(4^n) - 5(3^n)]\, u(n)$$

Linearity property states that:-

$$x(n) = a_1 x_1(n) + a_2 x_2(n).$$

$$= 3(4^n)u(n) - 5(3^n)u(n)$$

$$x_1(n) = 4^n u(n). \quad x_2(n) = 3^n u(n)$$

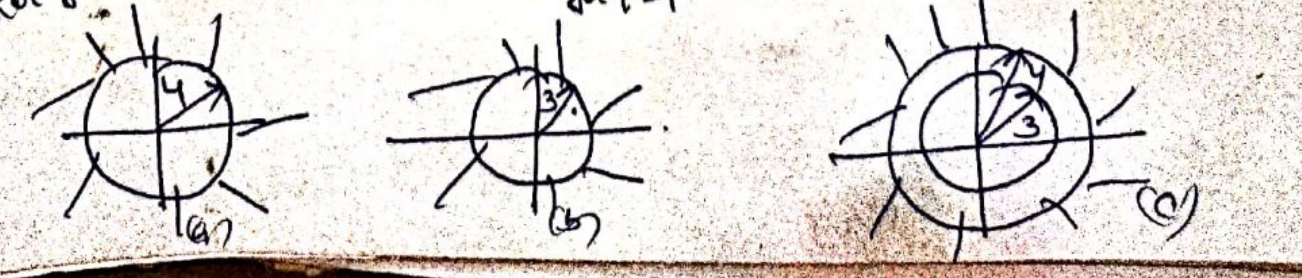
$$a^n u(n) \rightarrow \frac{Z}{Z-\alpha} \quad |Z| > |\alpha|$$

$$X_1(Z) = \frac{Z}{Z-4} \quad |Z| > 4$$

$$X_2(Z) = \frac{Z}{Z-3} \quad |Z| > 3$$

$$X(Z) = 3 \cdot \frac{Z}{Z-4} - 5 \times \frac{Z}{Z-3}$$

ROC for $|Z|>4$ — for $|Z|>3$ — combined ROC



## Time Shifting →

$$x(n) \xleftrightarrow{Z} X(Z)$$

$$x(n-K) \longleftrightarrow Z^{-K} X(Z)$$

Proof →

$$Z(x(n) \rightarrow X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z\, x(n-k) \rightarrow \sum_{n=-\infty}^{\infty} x(n-K) Z^{-n}$$

$Z^{-n}$ Can be written as $Z^{-n} = Z^{-(n-K)} \cdot Z^{-K}$

$$Z[x(n-k) \rightarrow \sum_{n=-\infty}^{\infty} x(n-k) Z^{-(n-K)} \cdot Z^{-K}$$

Now put $n-K=m$ on R.H.S. The limits will change as follow

$n=-\infty$, $-\infty-K=m=-\infty$

$n=\infty$, $\infty-K=m \Rightarrow m=\infty$

$$Z[x(n-k)] = Z^{-K} \cdot \sum_{m=-\infty}^{\infty} x(m) Z^{-m}$$

$$Z\, x(n-K) = Z^{-K} X(Z)$$

Similarly it Can be shown that

$$Zx(n+K) = Z^{+K} X(Z)$$

Example 4.22

* Find Z transform $x(n) = \delta(n+2)$

$\delta(n) \xleftrightarrow{Z} 1$ $\quad x(n+K) = Z^{K} X(z)$

$\delta(n+2) = Z^{2}$ Ans

Entire Z plane except $Z=\infty$

## Scaling Property →

$$x(n) \xleftrightarrow{n} X(Z) \qquad r_1 < |Z| < r_2$$

$$a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right) \qquad |a|r_1 < |Z| < |a| r_2$$

Proof →

$$Z(x(n)] = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z(a^n x(n) = \sum_{n=-\infty}^{\infty} a^n x(n) Z^{-n}$$

$$\sum_{n=-\infty}^{\infty} x(n) a^n Z^{-n} \Rightarrow \sum_{n=-\infty}^{\infty} x(n) (a^{-1} Z^{+1})^{-n}$$

$$Z\{a^n x(n) = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{Z}{a}\right)^{-n}$$

$$Z(a^n x(n) = X\left(\frac{Z}{a}\right)$$

$$a^n x(n) \xleftrightarrow{Z} X\left(\frac{Z}{a}\right)$$

ROC

$X(z)$ is $r_1 < |Z| < r_2$

replace $X(Z)$ by $X(Z/a)$ $\qquad r_1 < \left|\frac{Z}{a}\right| < r_2$

$$|a| r_1 < |Z| < |a| r_2$$

ROC of $X\left(\frac{Z}{a}\right) = |a| r_1 < |z| < |a| r_2$

Example 4.22

# Obtain Z transform of $x(n) = a^n u(n)$ using scaling property.

$$Z[u(n)] = \frac{Z}{Z-1} \quad \text{Roc } |Z|>1$$

According to scaling property

$$a^n x(n) \longleftrightarrow X\left(\frac{Z}{a}\right)$$

$$Z(a^n u(n)) \longleftrightarrow \frac{Z/a}{Z/a-1} \quad \text{Roc } \left|\frac{Z}{a}\right|>1$$

$$Z[a^n u(n)) = \frac{Z}{Z-a} \quad . \quad \text{Roc } |Z|>|a|$$

Example 4.24

# obtain Z transform of $x(n) = a^n \sin w_0 n\, u(n)$

Soln:- $$Z\{\sin n w_0 u(n)\} = \frac{Z \sin w_0}{Z^2 - 2Z\cos w_0 + 1} \quad |Z|>1$$

According to scaling property replace $Z$ by $\frac{Z}{a}$

$$\frac{\frac{Z}{\alpha}\sin w_0}{\frac{Z^2}{\alpha^2} - 2\frac{Z}{\alpha}\cos w_0 + 1}$$

divide Numinator & Denominator by $\alpha^2$

$$\frac{\alpha Z \sin w_0}{Z^2 - 2\alpha Z\cos w_0 + \alpha^2} \quad \left|\frac{Z}{a}\right|>1$$

$$|Z|>|a|$$

## Time Reversal Property →

If $x(n) \xleftrightarrow{Z} X(Z)$ $\qquad r_1 < |Z| < r_2$

$x(-n) \longleftrightarrow X Z^{-1}$ $\qquad \frac{1}{r_2} < |Z| < \frac{1}{r_1}$

Proof →

$$Z\{x(n)\} = X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}$$

$$Z[x(-n)] = \sum_{n=-\infty}^{\infty} x(-n) Z^{-n}$$

Put $l = -n$ the limit will change as

$n = -\infty$, $l = \infty$ $\quad$ when $n = \infty$, $l = -\infty$

$$Z[x(-n)] = \sum_{l=\infty}^{-\infty} x(l) Z^{l}$$

$$Z[x(-n)] = \sum_{l=-\infty}^{\infty} x(l) (Z^{-1})^{-l}$$

ROC

$$Z[x(-n)] = X(Z^{-1}) \qquad r_1 < \left|\frac{1}{Z}\right| < r_2$$

$|Z| < \frac{1}{r_1}$ & $|Z| > \frac{1}{r_2}$

$\frac{1}{r_2} < |Z| < \frac{1}{r_1}$

# $\cancel{x(n) = \left(\frac{1}{2}\right)^{n} u(-n)}$

Z ~~transform u(-n) =~~ $\cancel{\frac{1}{1-Z}}$

Example 4.25

# Z transform $x(n) = u(-n)$

$$Z[u(n)] = \frac{z}{z-1}$$

According to time reversal

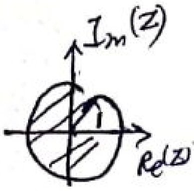
$$Z[x(-n)] = X(z^{-1})$$

Z transform of $x(-n)$ is obtain by replacing $z$ by $z^{-1}$

$$Z\,u(-n) = \frac{z^{-1}}{z^{-1}-1} \Rightarrow \frac{1}{1-z} \quad |z^{-1}|>1$$

ROC

$$Z(u(-n)) = \frac{1}{1-z} \quad \left|\frac{1}{z}\right|>1 \Rightarrow |1|>|z|$$




ROC is interior part of unit circle

## Differentiation in Z domain →

$$x(n) \xleftrightarrow{Z} X(z)$$

$$n\,x(n) \xleftrightarrow{Z} -z\frac{dX(z)}{dz}$$

Proof →

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

differentiate both sides with respect to $z$

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left[\sum_{n=-\infty}^{\infty} x(n) z^{-n}\right]$$

transfer $d/dz$ inside the summation

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} \frac{d}{dz}\left(x(n) z^{-n}\right) =$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n) \cdot x(n) z^{-n-1}$$

$$z^{-n-1} \Rightarrow z^{-n} \cdot z^{-1}$$

$$\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} (-n) x(n) z^{-n} z^{-1}$$

Take $z^{-1}$ outside the summation

$$\frac{dX(z)}{dz} = z^{-1} \sum_{n=-\infty}^{\infty} (-n) x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = -z^{-1} \sum_{n=-\infty}^{\infty} n \cdot x(n) z^{-n}$$

$$\frac{dX(z)}{dz} = -\frac{1}{z} \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} n x(n) z^{-n}$$

$$-z \frac{dX(z)}{dz} = Z\{n x(n)\}$$

$$n x(n) \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$$

Example 4.26 (10)

Obtain z transform using differentiation property

$x(n) = n a^n u(n)$

$x_1(n) = a^n u(n) \qquad x(n) = n x(n)$

According to differentiation property

$$n x_1(n) \xleftrightarrow{Z} -z \frac{d}{dz} X_1(z)$$

$$Z\{n a^n u(n)] \leftrightarrow -z \frac{d}{dz}\{a^n u(n)]$$

$$Z \text{ transform of } a^n u(n) = \frac{z}{z-a} \qquad |z|>|a|$$

$$Z[n a^n u(n)] \longrightarrow -z \frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z \frac{d}{dz}\left[z \cdot (z-a)^{-1}\right]$$

$$= -z\left\{z \cdot \frac{d}{dz}(z-a)^{-1} + (z-a)^{-1} \frac{d}{dz} z\right\}$$

$$-z\{z \cdot (z-a)^{-2} \cdot (-1) + (z-a)^{-1} \cdot 1$$

$$-z\left[\frac{-z}{(z-a)^2} + \frac{1}{(z-a)}\right] = -z\left[\frac{-z+z-a}{(z-a)^2}\right]$$

$$-z\left[\frac{-a}{(z-a)^2}\right] \Rightarrow \frac{az}{(z-a)^2} \qquad |z|>|a|$$

## Convolution of Two Sequences

If $x_1(n) \xleftrightarrow{Z} X_1(Z)$ and $x_2(n) \xleftrightarrow{Z} X_2(Z)$

$$x_1(n) * x_2(n) \longleftrightarrow X_1(Z) \cdot X_2(Z)$$

Proof: According to definition of convolution $x_1(n)$ & $x_2(n)$ can be written as:-

$$x(n) = x_1(n) \circledast x_2(n) \quad \text{--- (1)}$$

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \quad \text{--- (2)}$$

Taking Z transform of $x(n)$

$$X(Z) = Z\left\{ \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] \quad \text{--- (3)}$$

According to definition of Z transform

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) Z^{-n} \quad \text{--- (4)}$$

Put eq'n (3) in eq'n (4) and ~~...~~

$$Z[x(n)] = \sum_{n=-\infty}^{\infty} \left[ \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \right] Z^{-n}$$

Rearranging the summation

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left[ \sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-n} \right] \quad \text{--- (5)}$$

$Z^{-n} = Z^{-(n-k)}\, Z^{-k}$ can be written as:-

$$Z[x(n)] \left[ \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \right] \left[ \sum_{n=-\infty}^{\infty} x_2(n-k) Z^{-(n-k)} \right] \quad \text{--- (6)}$$

In second summation put $n-k=m$ Then limits will change as

$n = -\infty \Rightarrow -\infty - k = m \qquad m = -\infty$

$n = +\infty \Rightarrow \infty - k = m \qquad m = \infty$

$$Z[x(n)] = \left[ \sum_{k=-\infty}^{\infty} x_1(k) Z^{-k} \right] \cdot \left[ \sum_{m=-\infty}^{\infty} x_2(m) Z^{-m} \right] \quad \text{--- (7)}$$

$$Z[x(n)] = X_1(Z) \cdot X_2(Z) \Rightarrow x_1(n) \circledast x_2(n) \xleftrightarrow{Z} X_1(Z) \cdot X_2(Z)$$

Example 4.27



Find the linear Convolution of $x_1(n)$ and $x_2(n)$ (72)

$x_1(n) = \{2 \quad 3 \quad \underset{\uparrow}{4} \quad 1 \quad 2\}$

$x_2(n) = \{2 \quad \underset{\uparrow}{4} \quad 2 \quad 0 \quad 1\}$

Soln According to the property of linear convolution in z domain

$$Z\{x_1(n) \circledast x_2(n)\} = X_1(Z) \cdot X_2(Z)$$

Step 1

$x_1(n) = \{2, \ 3, \ \underset{\uparrow}{4}, \ 1, \ 2\}$

According to z transform

$$X_1(Z) = \sum_{n=-\infty}^{\infty} x_1(n) Z^{-n}$$

so the limits of summation are $n = -2$ to $2$

$$X_1(Z) = \sum_{n=-2}^{2} x_1(n) Z^{-n}$$

$$= x_1(-2) Z^{2} + x_1(-1) Z^{1} + x_1(0) Z^{0} + x_1(1) Z^{-1}$$

$$+ x_1(2) Z^{-2}$$

$$X_1(z) = 2Z^2 + 3Z^1 + 4z^0 + 1Z^{-1} + 2Z^{-2}$$

$$X_1(Z) = 2Z^2 + 3Z + 4 + \frac{1}{Z} + \frac{2}{Z^2} \quad -(1)$$

$[z^{-1}]$ + $z^{-1}$

ROC of $X_1(z)$

Put $Z = 0$ we get $0 + 0 + 4 + \frac{1}{0} + \frac{2}{0} \Rightarrow 0+0+4+\infty = \infty$

Put $Z = \infty$ we get $\infty + \infty + 4 + \frac{1}{\infty} + \frac{1}{\infty} \Rightarrow \infty + 4 + 0 = \infty$

ROC is entire Z plane except $Z=0$ and $Z=\infty$

Step 2

Now will obtain Z transform of $x_2(n)$

$$x_2(n) = \{2, \underset{\uparrow}{4}, 2, 0, 1\}$$

According to Z transform

$$X_2(Z) = \sum_{n=-\infty}^{\infty} x_2(n) Z^{-n}$$

So the limits of summation are $n = -1$ to $3$

$$X_2(Z) = \sum_{n=-1}^{3} x_2(n) Z^{-n}$$

$$= x_2(-1)Z^1 + x_2(0)Z^0 + x_2(1)Z^{-1} + x_2(2)Z^{-2} + x_2(3)Z^{-3}$$

$$= 2Z^1 + 4Z^0 + 2Z^{-1} + 0Z^{-2} + 1Z^{-3}$$

$$= 2Z + 4 + \frac{2}{Z} + \frac{1}{Z^3} \quad \text{--- (2)}$$

ROC of $X_2(Z)$

Put $Z=0$ we get $= 0 + 4 + \frac{1}{0} + \frac{1}{0} = 0 + 4 + \infty = \infty$

Put $Z=\infty$ we get $= \infty + 4 + \frac{1}{\infty} = \infty + 4 + 0 = \infty$

ROC is entire Z plane except $Z=0$ and $Z=\infty$

Step 3 $\quad X(Z) = X_1(Z) \cdot X_2(Z)$

$$X(Z) = (2Z^2 + 3Z^1 + 4 + 1Z^{-1} + 2Z^{-2})(2Z^1 + 4 + 2Z^{-1} + 0Z^{-2} + 1Z^{-3})$$

$$X(Z) = 4Z^3 + 8Z^2 + 4Z^1 + 2Z^{-1} + 6Z^2 + 12Z^1 + 6 + 3Z^{-2} +$$
$$8Z^1 + 16 + 8Z^{-1} + 4Z^{-3} + 2 + 4Z^{-1} + 2Z^{-2} + 1Z^{-4} + 4Z^{-1}$$
$$+ 8Z^{-2} + 4Z^{-3} + 2Z^{-5}$$